11/4/16

$(E, \langle\ \rangle)$ Ευκλείδιος Χώρος διάστασης n
Το σύνολο $\vec{\gamma_1}, \vec{\gamma_2}, \dots \vec{\gamma_n}$ ονομάζεται ορθοκανονική βάση αν $\langle \vec{\gamma_i}, \vec{\gamma_j} \rangle = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Διαδικασία Gram-Schmidt

$(E, \langle\ \rangle)$

$\vec{\alpha_1}, \vec{\alpha_2} \quad \vec{\alpha_n}$ Γραμμικά Ανεξάρτητα

$\vec{\beta_1}, \vec{\beta_2} \quad \vec{\beta_n}$ ορθογώνια

$\vec{\gamma_1}, \vec{\gamma_2} \quad \vec{\gamma_n}$ ορθοκανονικά

$\langle \vec{a_1} \rangle = \langle \vec{\beta_1} \rangle = \langle \vec{\gamma_1} \rangle$

$\langle \vec{a_1}, \vec{a_2} \rangle = \langle \vec{\beta_1}, \vec{\beta_2} \rangle = \langle \vec{\gamma_1}, \vec{\gamma_2} \rangle$

$\langle \vec{a_1}, \vec{a_2}, \ \vec{a_i} \rangle = \langle \vec{\beta_1}, \vec{\beta_2}, \ \vec{\beta_i} \rangle$
$= \langle \vec{\gamma_1}, \ \vec{\gamma_i} \rangle$
$1 \leq i \leq n$

Διαδικασία Gram-Schmidt

$\vec{\alpha_1} \quad \vec{a_n}$ Γ.Α.

Ορίζουμε $\vec{\beta_1} = \vec{a_1}$

$$\vec{\beta_2} = \vec{a_2} - \frac{\langle \vec{a_2}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1}$$

$$\vec{\beta_3} = \vec{a_3} - \frac{\langle \vec{a_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1} - \frac{\langle \vec{a_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \cdot \vec{\beta_2}$$

$$\vec{\beta_i} = \vec{a_i} - \frac{\langle a_i, \beta_1 \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1} \qquad - \frac{\langle \vec{a_i}, \vec{\beta_{i-1}} \rangle}{\langle \beta_{i-1}, \beta_{i-1} \rangle} \cdot \vec{\beta_{i-1}}$$

$$\vec{\beta_n} = \vec{a_n} - \frac{\langle \vec{a_n}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1} - \frac{\langle \vec{a_n}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \vec{\beta_2} \dots \frac{\langle \vec{a_n}, \vec{\beta_{n-1}} \rangle}{\langle \vec{\beta_{n-1}}, \vec{\beta_{n-1}} \rangle} \cdot \vec{\beta_{n-1}}$$

$$\vec{\gamma_1} = \frac{1}{\|\vec{\beta_1}\|} \cdot \vec{\beta_1}, \ \vec{\gamma_2} = \frac{1}{\|\vec{\beta_2}\|} \cdot \vec{\beta_2}, \qquad \vec{\gamma_n} = \frac{1}{\|\vec{\beta_n}\|} \vec{\beta_n}$$

Το σύνολο $\vec{\beta_1} \dots \vec{\beta_n}$ είναι ορθογώνιο

$\vec{\beta_1} \neq \vec{0}$

$\vec{\beta_1} \perp \vec{\beta_2}$

$$\langle \vec{\beta_1}, \vec{\beta_2} \rangle = \langle \vec{\beta_1}, \vec{\alpha_2} - \frac{\langle \vec{\alpha_2}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1} \rangle$$

$$= \vec{\beta_1}, -\frac{\langle \vec{\alpha_2}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1} \rangle$$

$$= \langle \vec{\beta_1}, \vec{\alpha_2} \rangle - \frac{\langle \vec{\alpha_2}, \vec{\beta_1} \rangle}{\cancel{\langle \vec{\beta_1}, \vec{\beta_1} \rangle}} \cancel{\langle \vec{\beta_1}, \vec{\beta_1} \rangle}$$

$$= 0$$

$\vec{\beta_2} \neq \vec{0}$ (γιατί αν $\vec{\beta_2} = 0$ τότε $\vec{\alpha_2} = \frac{\langle \vec{\alpha_2}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1} = \frac{\langle \vec{\alpha_2}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_2} \rangle} \vec{\alpha_1}$

$\vec{\alpha_1}, \vec{\alpha_2}$ Γ.Ε $\Rightarrow$ $\vec{\alpha_i}$ αι Γ.Ε. Άτοπο!)

$\vec{\beta_1} \neq 0$, $\vec{\beta_1} \perp \vec{\beta_2}$, $\vec{\beta_2} \neq 0$

$$\langle \vec{\beta_3}, \vec{\beta_1} \rangle = \langle \vec{\alpha_3} - \frac{\langle \vec{\alpha_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1} - \frac{\langle \vec{\alpha_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle}, \vec{\beta_1} \rangle$$

$$= \langle \vec{\alpha_3}, \vec{\beta_1} \rangle + \langle -\frac{\langle \vec{\alpha_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1}, \vec{\beta_1} \rangle + \langle -\frac{\langle \vec{\alpha_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \vec{\beta_2}, \vec{\beta_1} \rangle =$$

$$= \langle \vec{\alpha_3}, \vec{\beta_1} \rangle - \frac{\langle \vec{\alpha_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \langle \vec{\beta_1}, \vec{\beta_1} \rangle + \left(-\frac{\langle \vec{\alpha_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \right) \cancel{\langle \vec{\beta_2}, \vec{\beta_1} \rangle}^{0}$$

$$= \langle \vec{\alpha_3}, \vec{\beta_1} \rangle - \langle \vec{\alpha_3}, \vec{\beta_1} \rangle = 0$$ άρα $\vec{\beta_1} \perp \vec{\beta_3}$

$\langle \vec{\beta_3}, \vec{\beta_2} \rangle =$

$$\langle \vec{\beta_3}, \vec{\beta_2} \rangle = \left\langle \vec{a_3} - \frac{\langle \vec{a_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1} - \frac{\langle \vec{a_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \cdot \vec{\beta_2}, \vec{\beta_2} \right\rangle$$

$$= \langle \vec{a_3}, \vec{\beta_2} \rangle + \left\langle -\frac{\langle \vec{a_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \vec{\beta_1}, \vec{\beta_2} \right\rangle + \left\langle -\frac{\langle \vec{a_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \vec{\beta_2}, \vec{\beta_2} \right\rangle$$

$$= \langle \vec{a_3}, \vec{\beta_2} \rangle - \frac{\langle \vec{a_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \langle \vec{\beta_1}, \vec{\beta_2} \rangle - \frac{\langle \vec{a_3}, \vec{\beta_2} \rangle}{\cancel{\langle \vec{\beta_2}, \vec{\beta_2} \rangle}} \cdot \cancel{\langle \vec{\beta_2}, \vec{\beta_2} \rangle}$$

$$= \langle \vec{a_3}, \vec{\beta_2} \rangle - \langle \vec{a_3}, \vec{\beta_2} \rangle = 0$$

Ομοίως $\vec{\beta_1} \perp \vec{\beta_n}$

$\vec{\beta_2} \perp \vec{\beta_n}$

$\vec{\beta_{n-1}} \perp \vec{\beta_n}$

$\vec{\beta_n} \neq \vec{0}$

Άρα $\vec{\beta_1}, \dots, \vec{\beta_n}$ μη μηδενικά και ορθογώνια

Αφού $\vec{\beta_1}, \dots, \vec{\beta_n}$ ορθογώνια και μη μηδενικά (δηλ. $\|\vec{\beta_i}\| \neq 0$) τα $\vec{\gamma_1} = \frac{1}{\|\vec{\beta_1}\|} \vec{\beta_1}$, $\vec{\gamma_2} = \frac{1}{\|\vec{\beta_2}\|} \vec{\beta_2}$

$\vec{\gamma_n} = \frac{1}{\|\vec{\beta_n}\|} \vec{\beta_n}$ είναι ορθοκανονικά

Φροντιστηριακές ασκήσεις #5

Άσκηση 1.

$V \subseteq \mathbb{R}^4$

$V = \langle \vec{u_1}, \vec{u_2} \rangle \qquad \vec{u_1} = (1,1,0,0)$, $\vec{u_2} = (0,-1,0,2)$

$\vec{a_1} = \vec{u_1}, \ \vec{a_2} = \vec{u_2}$

$\vec{\beta_1} = \vec{a_1} = (1,1,0,0)$

$$\vec{\beta_2} = \vec{a_2} - \frac{\langle \vec{a_2}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1} = \frac{\langle (0,-1,0,2)(1,1,0,0) \rangle}{\langle (1,1,0,0)(1,1,0,0) \rangle} (1,1,0,0)$$

$$\vec{\beta_2} = (0,-1,0,2) - \frac{-1}{2}(1,1,0,0) = (0,-1,0,2) + \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right) =$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 0, 2\right)$$

Ελέγχω αν $\vec{\beta_1} \perp \vec{\beta_2}$ με $\langle \vec{\beta_1}, \vec{\beta_2} \rangle = 0$

$$\vec{\gamma_1} = \frac{1}{\|\vec{\beta_1}\|} \vec{\beta_1} = \frac{1}{\sqrt{2}}(1,1,0,0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$$

$$\vec{\gamma_2} = \frac{1}{\|\vec{\beta_2}\|} \cdot \vec{\beta_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + 0^2 + 2^2}} \left(\frac{1}{2}, -\frac{1}{2}, 0, 2\right) = \frac{1}{\sqrt{9/2}} \left(\frac{1}{2}, -\frac{1}{2}, 0, 2\right)$$

$$= \left(\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}, 0, \frac{2\sqrt{2}}{3}\right)$$

- Ορθογώνια βάση του V $\vec{\beta_1} = (1,1,0,0)$, $\vec{\beta_2} = \left(\frac{1}{2}, -\frac{1}{2}, 0, 2\right)$
- Ορθοκανονική βάση $\vec{\gamma_1} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right)$, $\vec{\gamma_2} = \left(\frac{\sqrt{2}}{6}, -\frac{\sqrt{2}}{6}, 0, \frac{2\sqrt{2}}{3}\right)$

$\vec{a_1} = (1,1,0,0)$

$\vec{a_2} = (0,-1,0,0)$ βάση του R^4

Τα εισάγαμε εμείς για να β. του R^4 $\left\{ \begin{array}{l} \vec{a_3} = (0,0,1,0) \\ \vec{a_4} = (0,0,0,1) \end{array} \right.$

$\vec{\beta_1} = (1,1,0,0)$

$\vec{\beta_2} = (1/2, 1/2, 0, 2)$

$$\vec{\beta_3} = \vec{a_3} - \frac{\langle \vec{a_3}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1} - \frac{\langle \vec{a_3}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \cdot \vec{\beta_2}$$

$$= (0,0,1,0) - \frac{\langle (0,0,1,0)(1,1,0,0) \rangle}{\langle (1,1,0,0)(1,1,0,0) \rangle}(1,1,0,0) - \frac{\langle (0,0,1,0)(\frac{1}{2},\frac{-1}{2},0,2) \rangle}{\langle (\frac{1}{2},\frac{-1}{2},0,2)(1/2,-\frac{1}{2},0,2) \rangle}(\frac{1}{2},\frac{-1}{2},0,2)$$

$$= (0,0,1,0) - 0(1,1,0,0) - 0(1/2,-1/2,0,2) = (0,0,1,0)$$

Άρα $\vec{\beta_3} = (0,0,1,0)$ (Ελέγχω αν $\vec{\beta_1} \perp \vec{\beta_2}$ και $\vec{\beta_3} \perp \vec{\beta_1}$)

$$\vec{\beta_4} = \vec{a_4} - \frac{\langle \vec{a_4}, \vec{\beta_1} \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle}\vec{\beta_1} - \frac{\langle \vec{a_4}, \vec{\beta_2} \rangle}{\langle \vec{\beta_2}, \vec{\beta_2} \rangle} \cdot \vec{\beta_2} - \cancel{\frac{\langle \vec{a_4}, \vec{\beta_3} \rangle}{\langle \vec{\beta_3}, \vec{\beta_3} \rangle} \vec{\beta_3}}$$

$$= (0,0,0,1) - \frac{\langle (0,0,0,1)(1,1,0,0) \rangle}{\langle \vec{\beta_1}, \vec{\beta_1} \rangle} \cdot \vec{\beta_1} -$$

$$\frac{\langle (0,0,0,1)(1/2,-1/2,0,2) \rangle}{(1/2,-1/2,0,2)(1/2,-1/2,0,2)}(1/2,-1/2,0,2) -$$

$$\frac{\langle (0,0,0,1)(0,0,1,0) \rangle}{\langle \vec{\beta_1}, \vec{\beta_2} \rangle} = (0,0,0,1) - \frac{2}{\frac{9}{2}}\left(\frac{1}{2}, -\frac{1}{2}, 0, 2\right) =$$

$= (0,0,0,1) - \frac{4}{9}\left(\frac{1}{2}, -\frac{1}{2}, 0, 2\right) = \left(-\frac{2}{9}, \frac{2}{9}, 0, \frac{1}{9}\right)$

$\langle \vec{\beta_2'}, \vec{\beta_4'} \rangle = \langle (1/2, -1/2, 0, 2), (-2/9, 2/9, 0, 1/9) \rangle$

$= -$

$\vec{\beta_1'}, \vec{\beta_2'}, \vec{\beta_3'}, \vec{\beta_4'}$: ορθογώνια βάση του $\mathbb{R}^4$

$\vec{j_1'} = \frac{1}{\|\vec{\beta_1'}\|} \cdot \vec{\beta_1'} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0\right)$

$\vec{j_2'} = \frac{1}{\|\vec{\beta_2'}\|} \cdot \vec{\beta_2'} = \left(\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, 0, \frac{2\sqrt{2}}{3}\right)$

$\vec{j_3'} = \frac{1}{\|\vec{\beta_3'}\|} \cdot \vec{\beta_3'} = \frac{1}{1}(0,0,1,0) = (0,0,1,0)$

$\vec{j_4'} = \frac{1}{\|\vec{\beta_4'}\|} \cdot \vec{\beta_4'} = \left(-\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{3}\right)$

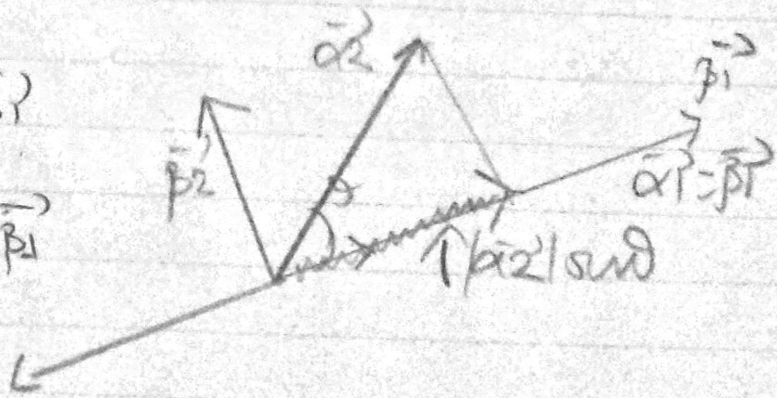
$|\vec{\beta_4'}| = \sqrt{\left(\frac{2}{9}\right)^2 + \left(\frac{2}{9}\right)^2 + 0^2 + \left(\frac{2}{9}\right)^2} = \sqrt{\frac{4}{81} + \frac{4}{81} + \frac{1}{81}} = \sqrt{\frac{9}{81}} = \frac{1}{3}$

$\vec{\alpha_1'}, \vec{\alpha_2'}$

$\vec{\beta_1'} = \vec{\alpha_1'}$

$\vec{\beta_2'} = \vec{\alpha_2'} - \frac{\langle \vec{\alpha_2'}, \vec{\beta_1'} \rangle}{\langle \vec{\beta_1'}, \vec{\beta_1'} \rangle} \cdot \vec{\beta_1'}$

$= \vec{\alpha_2'} \cdot \frac{|\vec{\alpha_2'}| \, |\vec{\beta_1'}| \, \sigma\upsilon\nu\theta}{|\vec{\beta_1'}|^2} \cdot \vec{\beta_1'}$



$\dfrac{\langle \bar{\alpha_2}, \bar{\beta_1} \rangle}{\langle \bar{\beta_1}, \bar{\beta_1} \rangle} \cdot \bar{\beta_1}$ προβολή του $\bar{\alpha_2}$ στον χώρο που παράγεται από το $\bar{\beta_1}$.

Ορισμός:
Έστω V υπόχωρος του Ευκλείδιου Χώρου Ε και $\bar{\beta_1}, \bar{\beta_2}, \ldots \bar{\beta_s}$ ορθογώνια βάση του V. Έστω $\bar{\alpha} \in E$ τότε η (ορθογώνια) προβολή του $\bar{\alpha}$ στον V είναι το διάνυσμα:

$$\Pi_V(\bar{\alpha}) = \frac{\langle \bar{\alpha}, \bar{\beta_1} \rangle}{\langle \bar{\beta_1}, \bar{\beta_1} \rangle} \cdot \bar{\beta_1} + \ldots + \frac{\langle \bar{\alpha}, \bar{\beta_s} \rangle}{\langle \bar{\beta_s}, \bar{\beta_s} \rangle} \cdot \bar{\beta_s} \in V$$

Το $\bar{\alpha} - \Pi_V(\bar{\alpha})$ είναι κάθετο σε κάθε διάνυσμα του V.

Ορισμός Έστω V υπόχωρος του Ευκλείδιου Χώρου Ε και $\bar{\gamma_1}, \bar{\gamma_2}, \ldots, \bar{\gamma_s}$ ορθοκανονική βάση του V. Έστω $\bar{\alpha} \in E$ τότε η (ορθογώνια) προβολή του $\bar{\alpha}$ στο V είναι το διάνυσμα:

$$\Pi_V(\bar{\alpha}) = \langle \bar{\alpha}, \bar{\gamma_1} \rangle \cdot \bar{\gamma_1} + \langle \bar{\alpha}, \bar{\gamma_2} \rangle \cdot \bar{\gamma_2} + \ldots + \langle \bar{\alpha}, \bar{\gamma_s} \rangle \cdot \bar{\gamma_s}$$